ON THE STABILITY CONDITIONS OF EQUILIBRIUM MAGNETOHYDRODYNAMIC CONFIGURATIONS

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In [1] it was shown by the direct method of Liapunov that an equilibrium state of an ideally conducting fluid is unstable, if there exist displacements of the fluid $\xi(\mathbf{r})$ from the equilibrium position, for which the potential energy of the system $U\{\xi\} < 0$. One naturally expects that if the potential energy of the system increases ($U\{\xi\} > 0$) for all admissible $\xi(\mathbf{r})$, then the equilibrium position is stable. However, using the definition of stability in [1], which does not restrict the derivatives $\partial\xi/\partial x_k$, it is impossible to prove this assertion. Taking the definition of stability in [1], there are admitted large perturbations of $\partial\xi/\partial x_k$, and particularly, large perturbations in the potential energy of the system. We observe that an analogous situation occurs in studying the stability of elastic systems [2].

In the present paper, we give a definition of the stability of equilibrium positions of ideally conducting fluids, different from the definition given in [1], and establish sufficient and necessary criteria for stability, which are close to the well-known criteria based on the energy principle [3-5].

1. The equations of motion for a small fluid displacement $\xi(\mathbf{r}, t)$ from the equilibrium position have the form [3-5]

$$\rho\ddot{\xi}_i = F_i(\xi) + f_i(\dot{\xi}) \qquad (i = 1, 2, 3) \tag{1.1}$$

where

$$\mathbf{F}(\xi) = \nabla(\xi\nabla p) + \gamma\nabla(p\operatorname{div}\xi) - \\ - \frac{1}{4\pi}\mathbf{H}\times\operatorname{rot}\operatorname{rot}(\xi\times\mathbf{H}) + \frac{1}{4\pi}\operatorname{rot}\mathbf{H}\times\operatorname{rot}(\xi\times\mathbf{H}) \tag{1.2}$$

$$f_i(\mathbf{v}) = \frac{\partial}{\partial x_k}\left[\eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)\right] + \frac{\partial}{\partial x_i}\left(\zeta\frac{\partial v_l}{\partial x_l}\right) \tag{1.3}$$

Here ρ, p and $\mathbf{H}$ are the equilibrium values of the fluid density, pressure, and the magnetic field intensity, γ is the adiabatic exponent, η and ζ are the viscosity coefficients, $\mathbf{v} = \dot{\xi}$ is the velocity of the fluid, and δ_{ik} is the Kronecker delta.

In order not to consider any perturbations of the magnetic field in the regions not occupied by the fluid, we assume that $\xi = 0$ on the surface S, which is the boundary of the volume V occupied by the fluid.

Let $\xi(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))$ denote the solution to Equation (1.1), satisfying the boundary conditions and the initial conditions $\xi = \xi_0(\mathbf{r})$, $\dot{\xi} = \dot{\xi}_0(\mathbf{r})$, at $t = 0$. Moreover, we shall assume that Equation (1.1) has solutions twice continuously differentiable with respect to x_k defined for all $t \geqslant 0$, if $\xi_0(\mathbf{r})$ and $\dot{\xi}_0(\mathbf{r})$ are twice continuously differentiable with respect to x_k.

We note that in what follows the initial data $\xi_0(\mathbf{r})$ and $\dot{\xi}_0(\mathbf{r})$ are always twice continuously differentiable, even though they are not so specified explicitly.

Then we find from (1.1) that

$$\frac{dE}{dt} = -W \leqslant 0 \qquad (E\{\xi, \dot{\xi}\} = T\{\dot{\xi}\} + U\{\xi\}) \tag{1.4}$$

(T is the kinetic energy, U the potential energy)

$$T\{\dot{\xi}\} = \frac{1}{2}\int \rho\dot{\xi}^2\, d\mathbf{r}, \qquad U\{\xi\} = -\frac{1}{2}\int \xi\mathbf{F}(\xi)\, d\mathbf{r} \tag{1.5}$$

$$W\{\mathbf{v}\} = -\int \mathbf{v}\mathbf{f}(\mathbf{v})\, d\mathbf{r} = \frac{1}{2}\int \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)^2 d\mathbf{r} + \int \zeta\left(\frac{\partial v_l}{\partial x_l}\right)^2 d\mathbf{r} \tag{1.6}$$

In the expressions (1.5) and (1.6) the integrations are taken over the volume V. The functional $U\{\xi\}$ contains ξ, $\partial\xi/\partial x_k$ $\partial^2\xi/\partial x_i\partial x_k$. However, using integration by parts and the fact that ξ vanishes on S, one readily sees that the second derivatives $\partial^2\xi/\partial x_i\partial x_k$ are not contained in $U\{\xi\}$. Obviously, $T(0) = U(0) = 0$

2. We denote

$$\rho_1\{\xi(\mathbf{r})\} = \int \xi^2\, d\mathbf{r} + \alpha\sum_{k=1}^{3}\int\left(\frac{d\xi}{dx_k}\right)^2 d\mathbf{r}, \qquad \rho_2\{\dot{\xi}(\mathbf{r})\} = \int \rho\dot{\xi}^2\, d\mathbf{r} \tag{2.1}$$

In (2.1), the integrations are carried out over the volume V, $\partial\xi/\partial x_k$

is the vector with coordinates $\partial\xi_i/\partial x_k$, and the coefficient α characteristic of the numbers of dimensions is a positive constant. We observe that under the integral sign defining ρ_1, we may introduce a positive weight function, selected from physical considerations.

We note that in what follows the numbers $\varepsilon > 0$ and $\delta > 0$ are always bounded above by some number $\sigma > 0$, even though not explicitly stated. We now give the definitions of stability and instability of an equilibrium position.

Definition 2.1. An equilibrium position is stable, if for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$, such that if

$$\rho_1\{\xi_0(\mathbf{r})\} < \delta_1, \qquad \rho_2\{\dot{\xi}_0(\mathbf{r})\} < \delta_2 \tag{2.2}$$

then for all $t \geqslant 0$

$$\rho_1\{\xi(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))\} < \varepsilon_1, \qquad \rho_2\{\dot{\xi}(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))\} < \varepsilon_2 \tag{2.3}$$

Definition 2.2. An equilibrium position is unstable, if there exists at least one of the numbers $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, so that for any $\delta_1 > 0$ and $\delta_2 > 0$ (however small), there are always such data

$$\xi_0(\mathbf{r}),\ \dot{\xi}_0(\mathbf{r}),\ \rho_1\{\xi_0(\mathbf{r})\} < \delta_1,\ \rho_2\{\dot{\xi}_0(\mathbf{r})\} < \delta_2$$

that at least one of the following inequalities

$$\rho_1\{\xi(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))\} \geqslant \varepsilon_1, \qquad \rho_2\{\dot{\xi}(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))\} \geqslant \varepsilon_2 \tag{2.4}$$

holds for at least one value of $t \geqslant 0$.

Definition 2.3. The functional $V\{\xi(\mathbf{r}), \dot{\xi}(\mathbf{r})\}$ is called positive-definite with respect to the metric $\rho\{\xi, \dot{\xi}\}$, if $V\{\xi, \dot{\xi}\} \geqslant 0$ for all admissible $\xi(\mathbf{r})$ and $\dot{\xi}(\mathbf{r})$, and if for any $\varepsilon > 0$, there exists a positive $\lambda = \lambda(\varepsilon)$ depending only on ε, such that $V\{\xi, \dot{\xi}\} \geqslant \lambda$ for any $\xi(\mathbf{r})$, $\dot{\xi}(\mathbf{r})$, satisfying the condition $\rho\{\xi(\mathbf{r}), \dot{\xi}(\mathbf{r})\} \geqslant \varepsilon$.

The functional $T\{\dot{\xi}(\mathbf{r})\}$ is positive-definite with respect to the metric $\rho_2\{\dot{\xi}(\mathbf{r})\}$.

3. We now consider the conditions of stability of equilibrium positions of ideally conducting inviscid fluids ($\eta = \zeta = 0$).

Theorem 3.1. (*Necessary condition for stability.*) In order for an equilibrium position of an ideally conducting inviscid fluid to be stable, it is necessary that $U\{\xi\} \geqslant 0$ for all admissible $\xi(\mathbf{r})$ (ξ twice continuously differentiable with respect to x_k and $\xi = 0$ on S).

The proof of this theorem is carried out in a similar way as the proof of Theorem 1 of [1].

Corollary. (*Sufficient condition for instability.*) From theorem 3.1, it follows that if there exists a $\xi(\mathbf{r})$ with $U\{\xi(\mathbf{r})\} < 0$, then the equilibrium position of the ideally conducting inviscid fluid is unstable.

Theorem 3.2. (*Sufficient condition for stability.*) If $U\{\xi\}$ is a positive-definite functional with respect to the metric $\rho_1\{\xi\}$, then the equilibrium position of an ideally conducting inviscid fluid is stable.

Proof. We select (any) $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, and show that we can find the corresponding (in the sense of Definition 2.1) $\delta_1 > 0$ and $\delta_2 > 0$.

Since $U\{\xi\}$ and $T\{\dot{\xi}\}$ are positive-definite functionals with respect to the metrics $\rho_1\{\xi\}$ and $\rho_2\{\dot{\xi}\}$, respectively, then for the given $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, we can find $\lambda_1(\varepsilon_1) > 0$, $\lambda_2(\varepsilon_2) > 0$, such that

$$U\{\xi(\mathbf{r})\} \geqslant \lambda_1, \qquad T\{\dot{\xi}(\mathbf{r})\} \geqslant \lambda_2 \quad \text{for } \rho_1\{\xi(\mathbf{r})\} \geqslant \varepsilon_1, \ \rho_2\{\dot{\xi}(\mathbf{r})\} \geqslant \varepsilon_2 \tag{3.1}$$

We set $\lambda = \min(\lambda_1, \lambda_2)$ and

$$V\{\xi, \dot{\xi}\} = T\{\dot{\xi}\} + U\{\xi\} \tag{3.2}$$

Since $V \to 0$ as $\rho_1\{\xi\} \to 0$, $\rho_2\{\dot{\xi}\} \to 0$, then for $\lambda > 0$, we find $\delta_1 > 0$ and $\delta_2 > 0$ ($\delta_1 \leqslant \varepsilon_1$, $\delta_2 \leqslant \varepsilon_2$), such that

$$V\{\xi, \dot{\xi}\} < \lambda \quad \text{for } \rho_1\{\xi\} < \delta_1, \ \rho_2\{\dot{\xi}\} < \delta_2 \tag{3.3}$$

We show that the $\delta_1 > 0$ and $\delta_2 > 0$ thus found correspond to the given $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ in the sense of Definition 2.1.

We assume the converse, i.e. that at some moment $t = \tau$, at least one of the following inequalities holds:

$$\rho_1\{\xi(\tau, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))\} \geqslant \varepsilon_1, \qquad \rho_2\{\dot{\xi}(\tau, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))\} \geqslant \varepsilon_2 \tag{3.4}$$

where

$$\rho_1\{\xi_0(\mathbf{r})\} < \delta_1, \qquad \rho_2\{\dot{\xi}_0(\mathbf{r})\} < \delta_2$$

Then the following inequality will be satisfied:

$$V\{\xi(\tau, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r})), \ \dot{\xi}(\tau, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))\} \geqslant \lambda \tag{3.5}$$

On the other hand, according to (1.4) for $t \geqslant 0$

$$V\{\xi(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r})), \ \dot{\xi}(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))\} = V\{\xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r})\} \tag{3.6}$$

since $W \equiv 0$. From this, we have

$$V\{\xi(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r})), \ \dot{\xi}(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))\} < \lambda \tag{3.7}$$

since

$$\rho_1\{\xi_0(\mathbf{r})\} < \delta_1, \qquad \rho_2\{\dot{\xi}_0(\mathbf{r})\} < \delta_2$$

The resulting contradiction shows that none of the inequalities (3.4) could hold. Consequently, for all $t \geqslant 0$

$$\rho_1\{\xi(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))\} < \varepsilon_1, \qquad \rho_2\{\dot{\xi}(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))\} < \varepsilon_2 \tag{3.8}$$

i.e. the equilibrium position is stable. Q.E.D.

We remark that in a similar manner it is possible to prove that $U\{\xi\} > 0$ gives a stable equilibrium position, with stability as defined in[1], if the initial perturbations are sufficiently smooth

$$(\,|\, \partial\xi_0 \,|\, \partial x_i \,| < \sigma_1, \,|\, \partial^2\xi_0 \,|\, \partial x_i\, \partial x_k \,| < \sigma_2,$$

where σ_1 and σ_2 are constants).

Theorem 3.2 is analogous to the wellknown theorem of Lagrange [6], the generalization of which was given by Movchan to study the stability [2] of elastic systems and solid bodies [7].

Obviously, an equilibrium position of an ideally conducting inviscid fluid cannot be asymptotically stable.

4. We now consider the case of a viscous ideally conducting fluid. We remark that the influence of viscosity on the stability of equilibrium positions of incompressible fluids has been considered by Hare using the method of normal waves [8].

Theorem 4.1. (*Sufficient condition for instability.*) If there exist $\xi(\mathbf{r})$ such that $U\{\xi(\mathbf{r})\} < 0$, then the equilibrium position is unstable with viscosity present.

Proof. We assume that the equilibrium state is stable. We choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$; then there exist $\delta_1 > 0$ and $\delta_2 > 0$, such that if

$$\rho_1\{\xi_0(\mathbf{r})\} < \delta_1, \qquad \rho_2\{\dot{\xi}_0(\mathbf{r})\} < \delta_2 \tag{4.1}$$

then

$$\rho_1\{\xi(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))\} < \varepsilon_1, \qquad \rho_2\{\dot{\xi}(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))\} < \varepsilon_2 \quad \text{for } t \geqslant 0 \tag{4.2}$$

We set

$$V\{\xi, \dot{\xi}\} = \int \rho\xi\dot{\xi}d\mathbf{r} + \frac{1}{4}\int \eta \left(\frac{\partial \xi_i}{\partial x_k} + \frac{\partial \xi_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial \xi_l}{\partial x_l}\right)^2 d\mathbf{r} + \frac{1}{2}\int \zeta \left(\frac{\partial \xi_l}{\partial x_l}\right)^2 d\mathbf{r} \tag{4.3}$$

where the integrations are carried out over the volume V.

If ξ is a solution to Equation (1.1), then

$$\frac{dV}{dt} = 2(T - U) \tag{4.4}$$

There exist $\xi_0{}^*(\mathbf{r})$ and $\dot{\xi}_0{}^*(\mathbf{r})$ such that

$$\rho\{\xi_0{}^*(\mathbf{r})\} < \delta_1, \quad \rho_2\{\dot{\xi}_0{}^*(\mathbf{r})\} < \delta_2 \tag{4.5}$$

$$\mu = -T\{\dot{\xi}_0{}^*(\mathbf{r})\} - U\{\xi_0{}^*(\mathbf{r})\} > 0, \qquad V\{\xi_0{}^*(\mathbf{r}), \dot{\xi}_0{}^*(\mathbf{r})\} > 0$$

Since for $t \geqslant 0$

$$\rho_1\{\xi(t, \mathbf{r}, \xi_0{}^*(\mathbf{r}), \dot{\xi}_0{}^*(\mathbf{r}))\} < \varepsilon_1, \qquad \rho_2\{\dot{\xi}(t, \mathbf{r}, \xi_0{}^*(\mathbf{r}), \dot{\xi}_0{}^*(\mathbf{r}))\} < \varepsilon_2 \tag{4.6}$$

then there exists a $\lambda > 0$, such that for $t \geqslant 0$

$$V\{\xi(t, \mathbf{r}, \xi_0{}^*(\mathbf{r}), \dot{\xi}_0{}^*(\mathbf{r})), \dot{\xi}(t, \mathbf{r}, \xi_0{}^*(\mathbf{r}), \dot{\xi}_0{}^*(\mathbf{r}))\} < \lambda \tag{4.7}$$

On the other hand, since for $t \geqslant 0$

$$T\{\dot{\xi}(t, \mathbf{r}, \xi_0{}^*(\mathbf{r}), \dot{\xi}_0{}^*(\mathbf{r}))\} + U\{\xi(t, \mathbf{r}, \xi_0{}^*(\mathbf{r}), \dot{\xi}_0{}^*(\mathbf{r}))\} \leqslant -\mu < 0 \tag{4.8}$$

we see that for $t \geqslant 0$

$$\frac{d}{dt}V\{\xi(t, \mathbf{r}, \xi_0{}^*(\mathbf{r}), \dot{\xi}_0{}^*(\mathbf{r})), \dot{\xi}(t, \mathbf{r}, \xi_0{}^*(\mathbf{r}), \dot{\xi}_0{}^*(\mathbf{r}))\} \geqslant 2\mu > 0 \tag{4.9}$$

Consequently, as $t \to +\infty$

$$V\{\xi(t, \mathbf{r}, \xi_0{}^*(\mathbf{r}), \dot{\xi}_0{}^*(\mathbf{r})), \dot{\xi}(t, \mathbf{r}, \xi_0{}^*(\mathbf{r}), \dot{\xi}_0{}^*(\mathbf{r}))\} \to +\infty \tag{4.10}$$

which contradicts (4.7). The resulting contradiction shows that the assumption that the equilibrium state is stable is false. Q.E.D.

Remark. Theorem 2 (sufficient condition for instability) in [1] is false, since an error has been made in its proof (from the inequality

$$\|\xi(t, \mathbf{r}, \xi_0^*(\mathbf{r}), \dot{\xi}_0^*(\mathbf{r}))\| < \varepsilon_1, \ \|\dot{\xi}(t, \mathbf{r}, \xi_0^*(\mathbf{r}), \dot{\xi}_0^*(\mathbf{r}))\| < \varepsilon_2 \quad \text{for } t \geqslant 0$$

does not follow inequality (15) of [1]).

However, the theorem remains correct if:

1) the definition of stability is changed to that made in the present paper, or

2) the definition of stability remains the same as in [1], but the additional assumption is made that $|\partial\xi_i/\partial x_k|$ $(i, k = 1, 2, 3)$ are smaller than some constant.

Corollary. (*Necessary condition for stability.*) In order for an equilibrium position of an ideally conducting viscous fluid to be stable, it is necessary that $U\{\xi(\mathbf{r})\} \geqslant 0$ for all admissible $\xi(\mathbf{r})$.

Theorem 4.2. (*Sufficient condition for stability.*) If $U\{\xi\}$ is a positive-definite functional with respect to the metric $\rho_1\{\xi\}$, then the equilibrium position is stable with viscosity present.

The proof of Theorem 4.2 is similar to that of Theorem 3.2 using the functional (3.2). Only instead of inequality (3.6), there will be the following inequality:

$$V\{\xi(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r})), \ \dot{\xi}(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))\} \leqslant V\{\xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r})\} \qquad (4.11)$$

and thus, as before, inequality (3.7) holds.

Theorems 4.1) and (4.2 are analogous to the well-known theorems of Kelvin [6] on the influence of dissipative forces on the stability of equilibrium states of systems of material points.

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